

Automorphism Groups of Cyclic p -gonal Pseudo-real Riemann Surfaces

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Abstract. In this article we prove that the full automorphism group of a cyclic p -gonal pseudo-real Riemann surface of genus g is either a semidirect product $C_n \rtimes C_p$ or a cyclic group, where p is a prime > 2 and $g > (p-1)^2$. We obtain necessary and sufficient conditions for the existence of a cyclic p -gonal pseudo-real Riemann surface with full automorphism group isomorphic to a given finite group. Finally we describe some families of cyclic p -gonal pseudo-real Riemann surfaces where the order of the full automorphism group is maximal and show that such families determine some real 2-manifolds embedded in the branch locus of moduli space.

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1 Introduction

A Riemann surface is called *pseudo-real* if it admits anticonformal automorphisms but no anticonformal involution. Pseudo-real Riemann surfaces

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appear in a natural way in the study of the moduli space \mathcal{M}_g^K of Riemann surfaces considered as Klein surfaces. The moduli space \mathcal{M}_g of Riemann surfaces of genus g is a two-fold branch covering of \mathcal{M}_g^K and the preimage of the branch locus consists of the Riemann surfaces admitting anticonformal automorphisms; they are either real Riemann surfaces (admitting anticonformal involutions, see for instance [N]) or pseudo-real Riemann surfaces.

Pseudo-real Riemann surfaces are those Riemann surfaces which are equivalent to their conjugate but the equivalence is not realized by an involution; therefore they admit anticonformal automorphisms of order greater than 2. The square of such automorphisms are not trivial conformal automorphisms, which means that the points in \mathcal{M}_g corresponding to pseudo-real Riemann surfaces are in the singular set (branch locus) of the orbifold $\mathcal{M}_g = \mathbb{T}_g/\text{Mod}_g$.

Pseudo-real Riemann surfaces were first studied in [E] and [S]. Hyperelliptic pseudo-real Riemann surfaces were considered in [S3], [BCn], and [BT] and pseudo-real Riemann surfaces with cyclic automorphism groups in [Et]. In [BCC] the existence of pseudo-real Riemann surfaces of every genus $g \geq 2$ is established and this is followed by a study of pseudo-real Riemann surfaces of genus 2 and 3; in [BCs] all the topological types of actions of the automorphisms groups for these two genera and genus 4 are described. In [H1] and [H2] the author finds explicit equations for non-hyperelliptic pseudo-real Riemann surfaces.

The p -gonal surfaces are cyclic p -fold coverings of the Riemann sphere and there is a great deal of interest in the study of the automorphism groups of these surfaces (see for instance [B], [BCI], [BHS], [Ko], [W]). The groups of (conformal and anticonformal) automorphisms of p -gonal real Riemann surfaces are obtained in [BCI], under the assumption that the p -gonal morphism is normal; for the conformal automorphisms in the non-normal case see [W].

In the present work, we study the full groups of (conformal and anticonformal) automorphisms of pseudo-real Riemann surfaces of genus g that are cyclic p -gonal, where p is a prime > 2 and $g > (p-1)^2$. In these conditions we establish that there are only two possible types of full automorphism groups of cyclic p -gonal pseudo-real Riemann surfaces: they are either cyclic or semidirect product of cyclic groups $C_n \rtimes C_p$ (Theorem 4). We also obtain necessary and sufficient conditions for the existence of cyclic p -gonal pseudo-real Riemann surfaces with given full automorphism group.

In section 4 we obtain the maximal order of the automorphism groups of pseudo-real cyclic p -gonal Riemann surfaces of genus $g > (p-1)^2$, when $p-1$ divides g .

Finally, in section 5, we apply our results to describe some 2-manifolds embedded in the branch locus of moduli space corresponding to cyclic p -gonal pseudo-real Riemann surfaces with maximal symmetry.

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2 Non-Euclidean crystallographic groups and Pseudo-real Riemann surfaces

A *non-Euclidean crystallographic group* (or *NEC group*) is a discrete group of isometries of the hyperbolic plane \mathbb{D} (we consider the unit disc model). We shall assume that an NEC group has compact orbit space. If Γ is such a group, its algebraic structure is determined by its signature

$$(h; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}). \quad (1)$$

The orbit space \mathbb{D}/Γ is a surface, possibly with boundary. The number h is called the *genus* of Γ and equals the topological genus of \mathbb{D}/Γ , while k is the number of its boundary components; the sign is $+$ or $-$ according to whether or not the surface \mathbb{D}/Γ is orientable. The integers $m_i \geq 2$ are called the *proper periods*, and represent the branch indices over interior points of \mathbb{D}/Γ in the natural projection $\pi : \mathbb{D} \rightarrow \mathbb{D}/\Gamma$. The bracketed expressions $(n_{i1}, \dots, n_{is_i})$, some or all of them may be empty (with $s_i = 0$), are called *period cycles* and represent the branchings over the i^{th} hole in the surface, and the numbers $n_{ij} \geq 2$ are named *link periods*.

There exists, associated with each signature, a canonical presentation for the group Γ , and a formula for the hyperbolic area of its fundamental domain (see [BEGG]). If the signature has sign $+$ then Γ has the following generators:

$$\begin{aligned} & x_1, \dots, x_r \quad (\text{elliptic transformations}), \\ & c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k} \quad (\text{reflections}), \\ & e_1, \dots, e_k \quad (\text{boundary transformations}), \\ & a_1, b_1, \dots, a_g, b_g \quad (\text{hyperbolic transformations}); \end{aligned}$$

and these generators satisfy the relations

$$\begin{aligned} & x_i^{m_i} = 1 \quad (\text{for } 1 \leq i \leq r), \\ & c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1, \quad c_{is_i} = e_i^{-1}c_{i0}e_i \quad (\text{for } 1 \leq i \leq k, 0 \leq j \leq s_i), \end{aligned}$$

$$x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_h b_h a_h^{-1} b_h^{-1} = 1 \text{ (long relation)}$$

If the sign is $-$ then we just replace the hyperbolic generators a_i, b_i by glide reflections d_1, \dots, d_h , and the long relation by $x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_h^2 = 1$.

The hyperbolic area of an arbitrary fundamental region of an NEC group Γ with signature (1) is

$$\mu(\Gamma) = 2\pi \left(\varepsilon h - 2 + k + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}} \right) \right)$$

where $\varepsilon = 2$ if the sign is $+$, and $\varepsilon = 1$ if the sign is $-$. Furthermore, any discrete group Λ of isometries of \mathbb{D} containing Γ as a subgroup of finite index is also an NEC group, and the hyperbolic area of a fundamental region for Λ is given by the Riemann-Hurwitz formula:

$$[\Lambda : \Gamma] = \mu(\Gamma) / \mu(\Lambda).$$

For any NEC group Λ , the *canonical Fuchsian subgroup* Λ^+ is its subgroup of orientation-preserving elements. If $\Lambda^+ \neq \Lambda$ then Λ^+ has index 2 in Λ , and we say that Λ is a *proper* NEC group.

Let X be a compact Riemann surface of genus $g > 1$. Then there is a surface Fuchsian group Γ , which is an NEC group with signature $(g; +; [-]; \{-\})$, such that $X = \mathbb{D}/\Gamma$, and the full (conformal and anticonformal) automorphism group $\text{Aut}(X)$ of X is isomorphic to Δ/Γ , where Δ is an NEC group normalizing Γ . We denote by $\text{Aut}^+(X)$ the group Δ^+/Γ of all conformal automorphisms of X .

Definition 1 *A Riemann surface is called pseudo-real if it admits anticonformal automorphisms but no anticonformal involution.*

In [BCC] we have established some basic results on the automorphism groups of pseudo-real Riemann surfaces

Proposition 2 [BCC] *Let X be a pseudo-real Riemann surface, and let G be the full automorphism group of X . Then 4 divides the order of G .*

Proposition 3 [BCC] *Suppose the pseudo-real surface X is conformally equivalent to \mathbb{D}/Γ , where Γ is a surface Fuchsian group which is normalized by an NEC group Δ such that $\Delta/\Gamma \cong G = \text{Aut}(X)$. Then the signature of Δ has the form $(h; -; [m_1, \dots, m_r])$ and the signature of the canonical Fuchsian subgroup Δ^+ of Δ is*

$$(h - 1; +; [m_1, m_1, m_2, m_2, \dots, m_r, m_r]).$$

3 Automorphism groups of cyclic p -gonal pseudo-real Riemann surfaces

In this section we establish our main theorem on the algebraic structure of the full automorphism groups of cyclic p -gonal pseudo-real Riemann surfaces:

Theorem 4 *Let X be a cyclic p -gonal (p is a prime > 2) pseudo-real Riemann surface of genus $g > (p-1)^2$. Let G be the full automorphism group of X , $H \cong C_p$ be the subgroup of G generated by a p -gonality automorphism and Δ be an NEC group uniformizing X/G . Hence the genus g of X must be even and the possible isomorphy types for the group G are:*

1. $G \cong C_n \rtimes_r H$, where 4 divides n , the first factor is generated by an anticonformal automorphism and

$$C_n \rtimes_r H = \langle x, y : x^p = 1; y^n = 1; y^{-1}xy = x^r \rangle \quad (2)$$

where $0 < r < p$, $r^n \equiv 1 \pmod{p}$. The NEC group Δ has signature either:

$$(1; -; [p, \frac{2(g+p-1)}{n(p-1)}, p, n/2]) \quad (3)$$

or, if in the presentation (2) $r = 1$ (direct product) or $r = p-1$, the signature of Δ may be also:

$$(1; -; [p, \frac{2g}{n(p-1)}, p, \frac{n}{2}p]) \quad (4)$$

2. $G \cong C_{np}$ (where 4 divides n) and G is generated by an anticonformal automorphism. The NEC group Δ has either signature (3) or (4).

Proof. Let Γ be a surface Fuchsian group uniformizing X , i. e. $X = \mathbb{D}/\Gamma$. There is an NEC group Δ and a Fuchsian group Λ such that $X/G \cong \mathbb{D}/\Delta$, $X/H \cong \mathbb{D}/\Lambda$ and

$$\Gamma \triangleleft \Lambda < \Delta; \Gamma \triangleleft \Delta; \Lambda/\Gamma \cong H; \Delta/\Gamma \cong G$$

The signature of Λ is

$$(0; +; [p, \frac{q}{p}, p])$$

where $q = \frac{2(g+p-1)p}{p-1}$, and by Proposition 3, the signature of Δ has the form

$$(h; -; [m_1, \dots, m_r])$$

Assume that $n = [\Delta : \Lambda]$. If the genus of X satisfies $g > (p-1)^2$, then the p -gonal covering $X \rightarrow \mathbb{C} = X/H$ is unique, $[A]$, hence $\Lambda \triangleleft \Delta$ and there is

$\varpi : \Delta \rightarrow \Delta/\Gamma \cong G$ such that $\varpi^{-1}(H) = \varpi^{-1}(\Lambda/\Gamma) = \Lambda$. Changing the indices in the canonical generators x_1, \dots, x_r in such a way that $\varpi(x_i^j) \notin \Lambda/\Gamma$, for any $0 < j < m_i$, $i = t+1, \dots, r$, but this condition is not satisfied for x_1, \dots, x_t , we obtain:

$$[m_1, \dots, m_r] = [ps_1, \dots, ps_t, m_{t+1}, \dots, m_r].$$

Given the signature of Λ , the fact that $n = [\Delta : \Lambda]$ implies that the integers s_1, \dots, s_t divide to n and $q = \frac{n}{s_1} + \dots + \frac{n}{s_t}$. Applying Riemann-Hurwitz formula we get:

$$\frac{-2 + q(1 - \frac{1}{p})}{h - 2 + \sum_{i=1}^t (1 - \frac{1}{s_i p}) + \sum_{i=t+1}^r (1 - \frac{1}{m_i})} = n \quad (5)$$

From the above formula and using $q = \sum \frac{n}{s_i}$ we obtain:

$$\begin{aligned} n(h-2) &= -2 + (\sum_{i=1}^t \frac{n}{s_i})(1 - \frac{1}{p}) - n \sum_{i=1}^t (1 - \frac{1}{s_i p}) - n \sum_{i=t+1}^r (1 - \frac{1}{m_i}) \\ &< (\sum_{i=1}^t \frac{n}{s_i})(1 - \frac{1}{p}) - n \sum_{i=1}^t (1 - \frac{1}{s_i p}) = n \sum_{i=1}^t (\frac{1}{s_i} - 1) \leq 0 \end{aligned}$$

Thus $h = 1$ (note that $h > 0$, since the sign in the signature of Δ is $-$). Now formula (5) is equivalent to:

$$2 - n + n \sum_{i=1}^t (1 - \frac{1}{s_i}) + n \sum_{i=t+1}^r (1 - \frac{1}{m_i}) = 0$$

where $s_i \geq 1$ and $m_i \geq 2$. From there it is easy to deduce that there is at most one $s_i > 1$. We have two possible cases:

i) $s_i = 1$, for all i , in which case we have

$$\sum_{i=t+1}^r (1 - \frac{1}{m_i}) = 1 - \frac{2}{n},$$

thus $r - t = 1$ and $m_{t+1} = \frac{n}{2}$.

ii) There is one i (we may suppose $i = t$) such that $s_t > 1$, then $s_t = \frac{n}{2}$, $s_1 = \dots = s_{t-1} = 1$ and $r = t$.

Therefore the possible signatures for Δ are:

$$\begin{aligned} i) & (1; -; [p, \frac{2(g+p-1)}{n(p-1)}, p, \frac{n}{2}]) \\ ii) & (1; -; [p, \frac{2g}{n(p-1)}, p, \frac{n}{2}p]) \end{aligned} \quad (6)$$

Assume then that Δ is an NEC group with one of the signatures in (6) and let

$$l = \frac{2(g+p-1)}{n(p-1)}$$

in signature i) and

$$l = \frac{2g}{n(p-1)}$$

in signature ii).

We shall study the epimorphism

$$\theta : \Delta \rightarrow \Delta/\Lambda$$

Let d, x_1, \dots, x_{l+1} be the generators of a canonical presentation of Δ . If $\pi : \Delta/\Gamma \rightarrow \Delta/\Lambda$ is the natural projection, then $\pi \circ \varpi = \theta$, and since $\varpi(x_i) \in \Lambda/\Gamma, i = 1, \dots, l$, we have that $\theta(x_1) = \dots = \theta(x_l) = 1$, so the group Δ/Λ is generated by $\theta(d)$ and $\theta(x_{l+1}) (= \theta(d)^{-2}$ by the long relation of NEC groups). Therefore the group Δ/Λ is cyclic of order n and $G \cong \Delta/\Gamma$ is a normal extension of a cyclic group $C_p \cong \Lambda/\Gamma$ by a cyclic group $C_n \cong \Delta/\Lambda$:

$$0 \rightarrow C_p \rightarrow G \xrightarrow{\pi} C_n \rightarrow 0$$

The covering $X/H \cong \mathbb{D}/\Lambda \rightarrow X/G \cong \mathbb{D}/\Delta$ is a cyclic covering with automorphism group $C_n \cong \Delta/\Lambda = \langle \theta(d) \rangle = \langle \tilde{y} \rangle$. The lift $y \in G$ of the automorphism \tilde{y} to $X = \mathbb{D}/\Gamma$ must have order n or np . In the first case G is a semidirect product $C_n \ltimes_r H$ with presentation (2) and in the second case G is cyclic ($\cong C_{np}$).

Suppose the signature of Δ is *ii*) of (6).

Now $\varpi(x_{l+1})$ is an element of order $q = \frac{n}{2}p$ in G and $\varpi(d)$ has order either n or np .

If the order of $\varpi(d)$ is np , then $G \cong C_{np}$.

In the case $\varpi(d)$ has order n , G is generated by x, y where $\varpi(d) = y$, $\varpi(x_l) = x$, and the following relations hold:

$$x^p = 1; y^n = 1; y^{-1}xy = x^r$$

Since there is an element of order $\frac{n}{2}p$ in G then Δ^+/Γ is cyclic of order $\frac{n}{2}p$. Since $y^2 \in \Delta^+/\Gamma$, we have $y^{-2}xy^2 = x = x^{r^2}$, and then $r = 1$ or $p-1$.

In all the cases, the surface X being pseudo-real, we know by Proposition 2, that 4 divides n . Finally, either $\frac{2(g+p-1)}{n(p-1)}$ or $\frac{2g}{n(p-1)}$ is an integer, thus g is even. ■

Remarks. 1. Condition $g > (p-1)^2$ in the above theorem may be replaced by the assumption that the cyclic group H is normal in G . The n -gonal surfaces whose full automorphism group G is not the normalizer of H have been studied in [BW], [W]

2. Theorem 4 remains valid if G is a group of automorphisms (not necessarily the full automorphism group) containing a p -gonal automorphism and an anticonformal automorphism of order > 2 .

Theorem 5 *For each prime $p > 2$ and each n , such that 4 divides n , there exists a cyclic p -gonal pseudo-real Riemann surface of genus g with full automorphism group isomorphic to $C_p \rtimes_r C_n$ with presentation (2) and $1 < r < p-1$ if and only if $\frac{2(g+p-1)}{n(p-1)}$ is an integer > 1 .*

Proof. Assume that $l = \frac{2(g+p-1)}{n(p-1)}$ is an integer > 1 . Let Δ be a maximal NEC group with signature $(1; -; [p, \dots, p, \frac{n}{2}])$. We define

$$\varpi : \Delta \rightarrow C_n \rtimes_r C_p = \langle x, y : x^p = 1; y^n = 1; y^{-1}xy = x^r, r^n \equiv 1 \pmod{p} \rangle$$

by

$$\begin{aligned} \varpi(x_{2i+1}) &= x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \dots, (l-2)/2, \\ \varpi(x_{l+1}) &= y^2, \varpi(d) = y^{-1}, \text{ if } l \text{ is even} \end{aligned}$$

and

$$\begin{aligned} \varpi(x_{2i+1}) &= x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \dots, (l-3)/2, \\ \varpi(x_{l-2}) &= x, \varpi(x_{l-1}) = x, \varpi(x_l) = x^{-2} \\ \varpi(x_{l+1}) &= y^2, \varpi(d) = y^{-1}, \text{ if } l \text{ is odd, } l > 1 \end{aligned}$$

Every element of $C_n \rtimes_r C_p$ can be expressed as $x^\alpha y^\beta$, by using the relation $y^{-1}xy = x^r$. Note that $\varpi(\Delta^+) = \varpi \langle x_1, \dots, x_l, x_{l+1}, d^2 \rangle = \langle x, y^2 \rangle$, 4 divides n and $n/2$ is an even number. Thus the elements $x^\alpha y^{n/2}$ of order 2 of the group $C_n \rtimes_r C_p$, that can be written as $x^\alpha y^{n/2}$, correspond to orientation preserving automorphisms. The epimorphism ϖ cannot be extended since Δ is maximal and $\mathbb{D}/\ker \varpi$ is the pseudo-real Riemann surface that we are looking for.

If $l = 1$ by Theorem 2.4.7 of [BEGG] the group Δ is contained in an NEC group Ω with signature $(0; +; [2]; \{(p, \frac{n}{2})\})$. Let

$$\begin{aligned} C_2 \rtimes (C_n \rtimes_r C_p) &= \\ \langle s, x, y : s^2 = 1, x^p = 1; y^n = 1; y^{-1}xy = x^r; sxs = x^{-1}, sys = y^{-1} \rangle \end{aligned}$$

We may assume that $\varpi : \Delta \rightarrow C_n \rtimes_r C_p$ is given by $\varpi(d) = y, \varpi(x_1) = x, \varpi(x_2) = y^{-2}$, where d, x_1 and x_2 are generators of a canonical presentation of Δ . The epimorphism ϖ is the restriction to Δ of

$$\theta^* : \Omega \rightarrow C_2 \rtimes (C_n \rtimes C_p)$$

$$\theta^*(x'_1) = sx^\alpha y^{-1}, \theta^*(c'_0) = s, \theta^*(c'_1) = sx, \theta^*(c'_2) = sxy^{-2}, \theta^*(e) = sx^\alpha y^{-1}$$

where $\alpha \equiv \frac{p+1}{2}r^{n-1} \pmod{p}$. Hence there are anticonformal involutions in $\text{Aut}(\mathbb{D}/\ker \varpi)$ and $\mathbb{D}/\ker \varpi$ is not a pseudo-real Riemann surface. ■

Theorem 6 *For each prime $p > 2$ and each n , such that 4 divides n , there exists a cyclic p -gonal pseudo-real Riemann surface of genus g , with full automorphism group isomorphic to C_{np} if and only if either $\frac{2(g+p-1)}{n(p-1)}$ or $\frac{2g}{n(p-1)}$ are integers > 1 and $\gcd(p, n/2) = 1$.*

Proof. If either $\frac{2(g+p-1)}{n(p-1)}$ or $\frac{2g}{n(p-1)}$ are integers > 1 with $\gcd(p, n/2) = 1$, we may consider maximal NEC groups with signatures either

$$(1; -; [p, \frac{2(g+p-1)}{n(p-1)}, p, \frac{n}{2}]) \text{ or } (1; -; [p, \frac{2g}{n(p-1)}, p, \frac{n}{2}p])$$

respectively. The surface Fuchsian groups uniformizing the cyclic p -gonal pseudo-real Riemann surfaces having cyclic automorphism group are the kernel of the epimorphisms given by Theorem 4 of [Et].

Assume that there is a cyclic p -gonal pseudo-real Riemann surface X of genus g , with full automorphism group C_{np} . By Theorem 4, there is a maximal NEC group Δ with signature as given in (6) (then either $\frac{2(g+p-1)}{n(p-1)}$ or $\frac{2g}{n(p-1)}$ are integers) and there is an epimorphism $\varpi : \Delta \rightarrow C_{np}$ such that $\ker \varpi$ uniformizes X . Condition ii) of Theorem 4 of [Et] implies $\gcd(p, n/2) = 1$.

For the first possible signature of Δ , assume $\frac{2(g+p-1)}{n(p-1)} = 1$. The epimorphism $\varpi : \Delta \rightarrow C_{np} = \langle t \rangle$, where Δ has signature $(1; -; [p, \frac{n}{2}])$ is given by $\varpi(d) = t, \varpi(x_1) = t^{2m}, \varpi(x_2) = t^{-2m-2}$, where d, x_1 and x_2 are generators of a canonical presentation of Δ , and $0 < m < p$ (note that x_1 is an orientation preserving transformation and then $\varpi(x_1)$ must be an even power of t). By Theorem 2.4.7 of [BEGG] the group Δ is contained in an NEC group Ω with signature $(0; +; [2]; \{(p, \frac{n}{2}p)\})$ and the epimorphism ϖ is the restriction to Δ of

$$\theta^* : \Omega \rightarrow D_{np} = \langle s, t : s^2 = t^{np} = 1, sts = t^{-1} \rangle$$

$$\theta^*(x'_1) = t^{2m+1}, \theta^*(c'_0) = t^{2m}s, \theta^*(c'_1) = s, \theta^*(c'_2) = st^{-2m-2}$$

where x'_1, c'_0, c'_1, c'_2 is a set of generators of a canonical presentation of Ω . Then $\ker \theta^* = \ker \varpi = \Gamma$ and $\text{Aut}^\pm(\mathbb{D}/\Gamma)$ contains D_{np} with anticonformal involutions. Hence \mathbb{D}/Γ is not pseudo-real. Thus $\frac{2(g+p-1)}{n(p-1)} > 1$.

The argument for the second possible signature of Δ with $\frac{2g}{n(p-1)} = 1$ is similar. ■

Theorem 7 *For each prime $p > 2$ and each n , such that 4 divides n , there exists a cyclic p -gonal pseudo-real Riemann surface of genus g with full automorphism group isomorphic to $C_p \rtimes_r C_n$ with presentation (2) and $r = 1$ or $p-1$ if and only if either $\frac{2(g+p-1)}{n(p-1)}$ is an integer > 1 or $\frac{2g}{n(p-1)}$ is an integer > 1 and $\gcd(p, n/2) = 1$.*

Proof. If either $\frac{2(g+p-1)}{n(p-1)}$ or $\frac{2g}{n(p-1)}$ are integers > 1 , we may consider maximal NEC groups with signatures either

$$(1; -; [p, \frac{2(g+p-1)}{n(p-1)}, p, \frac{n}{2}]) \text{ or } (1; -; [p, \frac{2g}{n(p-1)}, p, \frac{n}{2}p])$$

respectively. For the first case considering the epimorphism defined in Theorem 5 we obtain the surface that we are looking for. If the signature is $(1; -; [p, \dots, p, \frac{n}{2}p])$, with $l = \frac{2g}{n(p-1)}$, and $\gcd(p, n/2) = 1$, then we define:

$$\varpi : \Delta \rightarrow C_n \rtimes_r C_p = \langle x, y : x^p = 1; y^n = 1; y^{-1}xy = x^r \rangle$$

by

$$\begin{aligned} \varpi(x_{2i+1}) &= x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \dots, (l-3)/2, \\ \varpi(x_l) &= x, \varpi(x_{l+1}) = x^{-1}y^2, \varpi(d) = y^{-1}, \text{ if } l \text{ is odd} \end{aligned}$$

and

$$\begin{aligned} \varpi(x_{2i+1}) &= x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \dots, (l-4)/2, \\ \varpi(x_{l-1}) &= x, \varpi(x_l) = x \\ \varpi(x_{l+1}) &= x^{-2}y^2, \varpi(d) = y^{-1}, \text{ if } l \text{ is even} \end{aligned}$$

Note that $x^{-1}y^2$ and $x^{-2}y^2$ have order $\frac{n}{2}p$ as consequence of the condition $\gcd(p, n/2) = 1$.

For the first possible signature of Δ , if we assume $\frac{2(g+p-1)}{n(p-1)} = 1$, the argument in Theorem 5 shows that the surfaces in these conditions are not pseudoreal. Assume now that Δ has the second possible signature with

$l = 1$. We may assume that the epimorphism $\varpi : \Delta \rightarrow C_p \rtimes_r C_n$, where Δ has signature $(1; -; [p, \frac{n}{2}p])$ is given by $\varpi(d) = y, \varpi(x_1) = x, \varpi(x_2) = x^{-1}y^{-2}$, where d, x_1 and x_2 are generators of a canonical presentation of Δ . By Theorem 2.4.7 of [BEGG] the group Δ is contained in an NEC group Ω with signature $(0; +; [2]; \{(p, \frac{n}{2}p)\})$. If $r = 1$, the epimorphism ϖ is the restriction to Δ of

$$\begin{aligned} \theta^* : \Omega &\rightarrow C_2 \rtimes (C_n \rtimes C_p) \\ \theta^*(x'_1) &= sy^{-1}, \theta^*(c'_0) = s, \theta^*(c'_1) = sx, \theta^*(c'_2) = sy^{-2}, \theta^*(e') = sy^{-1} \end{aligned}$$

where

$$\begin{aligned} C_2 \rtimes (C_n \rtimes_r C_p) = \\ \langle s, x, y : s^2 = 1, x^p = 1, y^n = 1, y^{-1}xy = x^r, sxs = x^{-1}, sys = y^{-1} \rangle, \\ r = 1, p - 1 \end{aligned}$$

Hence the surface uniformized by $\ker \varpi$ has anticonformal involutions.

■

4 Cyclic p -gonal pseudo-real Riemann surfaces with automorphism group of maximal order for a fixed genus.

For each type of automorphism groups of pseudo-real p -gonal Riemann surfaces of genus g , next proposition determines its maximal order:

Proposition 8 *Let p be a prime > 2 and $g > (p - 1)^2$. Assume that $p - 1$ divides g .*

1. *If $\frac{g}{p-1} \equiv 3 \pmod{4}$ there are pseudo-real cyclic p -gonal Riemann surfaces of genus g with full automorphism group of order $\frac{p(g+p-1)}{p-1}$ and this order is maximal for pseudo-real cyclic p -gonal Riemann surfaces of genus g . The full automorphism group is either isomorphic to a semidirect product of cyclic groups or, if $\gcd(p, \frac{g+p-1}{2(p-1)}) = 1$, may be isomorphic to a cyclic group.*

2. *If $\frac{g}{p-1} \equiv 0 \pmod{4}$ and $\gcd(p, \frac{g+p-1}{2(p-1)}) = 1$ there are pseudo-real cyclic p -gonal Riemann surfaces of genus g with full automorphism group of order $\frac{pg}{(p-1)}$ and this order is maximal for pseudo-real cyclic p -gonal Riemann surfaces of genus g . In this case the full automorphism group is either cyclic or $C_p \rtimes_r C_n$ with $r = 1$ or $p - 1$.*

Proof. Assume that X is a pseudo-real cyclic p -gonal Riemann surface of genus g with $X/\text{Aut}(X)$ uniformized by an NEC group where signature $i)$ of (6) in theorem 4 and then

$$2 \leq \frac{2(g+p-1)}{n(p-1)}$$

or equivalently:

$$n \leq \frac{g}{(p-1)} + 1$$

When $p-1$ divides g we may have $n = \frac{g}{(p-1)} + 1$. Hence there are surfaces with automorphism groups of order $np = \frac{p(g+p-1)}{p-1}$ and this order is maximal. Note that $\frac{g}{p-1} \equiv 3 \pmod{4}$, since 4 must divide n . By theorem 4 the automorphism group is either isomorphic to $C_p \rtimes_r C_n$ or, if $\gcd(p, \frac{g+p-1}{2(p-1)}) = 1$, may be isomorphic to C_{np} .

Now assume that X is a pseudo-real cyclic p -gonal Riemann surface of genus g with $X/\text{Aut}(X)$ uniformized by an NEC group with signature $ii)$ of (6).

Since 4 must divide n , if $\frac{g}{p-1} \equiv 0 \pmod{4}$ and $\gcd(p, \frac{g}{2(p-1)}) = 1$, there are surfaces with full automorphism group isomorphic to a maximal order group (the order is $\frac{pg}{2(p-1)}$). Note that when $\frac{g}{p-1} \equiv 0 \pmod{4}$, 4 does not divide $\frac{2(g+p-1)}{p-1}$ and then case 1 of theorem 4 is not possible. By theorem 4 case 2, the automorphism group is isomorphic to C_{np} or $C_p \rtimes_r C_n$ with $r = 1, p-1$. ■

5 Cyclic p -gonal pseudo-real Riemann surfaces in the moduli space.

In this section we apply the previous results to the study of branch locus of moduli spaces.

First we define:

$\mathcal{M}_g^{PR}(n, p) = \{X : X \text{ is a cyclic } p\text{-gonal pseudo-real Riemann surface of genus } g \text{ and with full automorphism group of order } np\}$.

Let \mathcal{M}_g be the moduli space of Riemann surfaces of genus g , \mathbb{T}_g be the corresponding Teichmüller space and

$$\pi : \mathbb{T}_g \rightarrow \mathbb{T}_g / \text{Mod}_g = \mathcal{M}_g$$

be the natural projection.

Proposition 9 *Let p be a prime > 2 and $g \geq 2$. Assume that $p-1$ divides g .*

1. *If $\frac{g}{p-1} \equiv 3 \pmod{4}$ then the set $\mathcal{M}^{PR}(\frac{g+p-1}{p-1}, p)$ is a 2-manifold in the branch locus of the orbifold \mathcal{M}_g .*
2. *If $\frac{g}{p-1} \equiv 0 \pmod{4}$ and $\gcd(p, \frac{g+p-1}{2(p-1)}) = 1$ then the set $\mathcal{M}^{PR}(\frac{g}{p-1}, p)$ is a 2-manifold in the branch locus of the orbifold \mathcal{M}_g .*

Proof. First we consider the case $\frac{g}{p-1} \equiv 3 \pmod{4}$ and we write $n = \frac{g}{p-1} + 1$. Let $\mathbb{T}_{(1; -; [p, p, n/2])}$ be the Teichmüller space for NEC groups with signature $(1; -; [p, p, \frac{n}{2}])$ and M be the subset of maximal groups of $\mathbb{T}_{(1; -; [p, p, n/2])}$.

Consider the set $S = \{(\sigma, i) : \sigma \text{ is a signature of NEC groups and } i \text{ is an inclusion of a group of signature } (1; -; [p, p, \frac{n}{2}]) \text{ in a group of signature } \sigma\}$. Using Riemann-Hurwitz and canonical presentations of NEC groups it is possible to show that the set S is finite. Each element $(\sigma, i) \in S$ produces an embedding $i_* : \mathbb{T}_\sigma \rightarrow \mathbb{T}_{(1; -; [p, p, n/2])}$ and $i_*(\mathbb{T}_\sigma)$ is closed (see [MS]). Then $M = \mathbb{T}_{(1; -; [p, p, n/2])} - \bigcup_{(\sigma, i) \in S} i_*(\mathbb{T}_\sigma)$ is an open set.

Let G be $C_p \rtimes_r C_n$, \mathcal{D} be an abstract group with presentation

$$\langle d, x_1, x_2, x_3 : d^2 x_1 x_2 x_3 = 1, x_1^p = x_2^p = x_3^{n/2} = 1 \rangle$$

and $\varpi : \mathcal{D} \rightarrow G$ be an epimorphism as considered in the proof of theorem 4. If Δ has signature $(1; -; [p, p, \frac{n}{2}])$ and $\eta : \Delta \rightarrow \mathcal{D}$ is the isomorphism given by the canonical presentation then $\ker \varpi \circ \eta$ uniformizes a pseudo-real cyclic p -gonal Riemann surface of genus g . The epimorphism ϖ induces an embedding $\mathbb{T}_{(1; -; [p, p, n/2])} \xrightarrow{\varpi^*} \mathbb{T}_g$ defined by $[\Delta] \mapsto [\ker \varpi \circ \eta]$ (see [MS]). For two embeddings ϖ_1^* and ϖ_2^* defined by two epimorphisms ϖ_1 and ϖ_2 , we have that $\varpi_1^*(M) \cap \varpi_2^*(M) = \emptyset$, since a possible point in the intersection would admit two different actions of maximal order which is absurd. Finally the action of the modular group Mod_g is fixed point free on $\pi^{-1}(\mathcal{M}^{PR}(n, p))$, which implies the projection

$$\pi : \pi^{-1}(\mathcal{M}^{PR}(n, p)) = \bigcup_{\varpi} \varpi^*(M) \rightarrow \mathcal{M}^{PR}(n, p)$$

is a local homeomorphism. Hence $\mathcal{M}^{PR}(n, p)$ is a manifold of (real) dimension 2 in \mathcal{M}_g .

The argument for the second case is similar.

■

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